EQUIVALENCE OF ROBUST STABILIZATION AND ROBUST PERFORMANCE VIA FEEDBACK

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ABSTRACT. One approach to robust control for linear plants with structured uncertainty as well as for linear parameter-varying (LPV) plants (where the controller has on-line access to the varying plant parameters) is through linear-fractional-transformation (LFT) models. Control issues to be addressed by controller design in this formalism include robust stability and robust performance. Here robust performance is defined as the achievement of a uniform specified L^2-gain tolerance for a disturbance-to-error map combined with robust stability. By setting the disturbance and error channels equal to zero, it is clear that any criterion for robust performance also produces a criterion for robust stability. Counter-intuitively, as a consequence of the so-called Main Loop Theorem, application of a result on robust stability to a feedback configuration with an artificial full-block uncertainty operator added in feedback connection between the error and disturbance signals produces a result on robust performance. The main result here is that this performance-to-stabilization reduction principle must be handled with care for the case of dynamic feedback compensation: casual application of this principle leads to the solution of a physically uninteresting problem, where the controller is assumed to have access to the states in the artificially-added feedback loop. Application of the principle using a known more refined dynamic-control robust stability criterion, where the user is allowed to specify controller partial-state dimensions, leads to correct robust-performance results. These latter results involve rank conditions in addition to Linear Matrix Inequality (LMI) conditions.

1. INTRODUCTION

Linear-Fractional-Transformation (LFT) models have been used for the study of stability issues for systems with structured uncertainty [17, 6], of robust gain-scheduling for Linear Parameter-Varying (LPV) systems [18, 17, 1, 11], and of model reduction for systems having structured uncertainty [7, 4, 15, 5]. It turns out that the LMI solution of the H^∞-control problem generalizes nicely to these more general structures; we refer the reader to the books [13, 10] for nice expositions of these and other related developments.

The results in the paper [17] (see also [16]) focus on synthesis of controllers implementing a somewhat stronger notion of stability known as $\mathcal{Q}$-stability. The notion of $\mathcal{Q}$-stability implies robust stability but the converse holds only for special structures (see [17]). One such special structure is the case where one allows the structured uncertainty to be time-varying (and perhaps also causal and/or slowly

time-varying in a precise sense—see [19, 3]); then $\mathcal{Q}$-stability is equivalent to robust stability with respect to this enlarged uncertainty structure. This observation gives perhaps the most compelling system-theoretic interpretation of Q-stability. Even when one is not working with this enlarged uncertainty structure, $\mathcal{Q}$-stability is still attractive since it is sufficient for robust stability and can be characterized in LMI form.

One result in [17] is a characterization of the existence of a static output feedback controller implementing $\mathcal{Q}$-stability in terms of the existence of positive-definite solutions X, Y to a pair of LMIs; the additional coupling condition $Y = X^{-1}$ destroys the convex character of the solution criterion and thereby makes the solution criterion computationally unattractive. A second result provides an LMI characterization for the existence of a dynamic (in the sense of multidimensional linear systems) controller and provides a Youla parametrization for the set of all such controllers. The question of the existence of controllers for LFT-model systems achieving $\mathcal{Q}$-performance (a scaled version of robust performance) is settled in [18, 11, 1] (see the book [10] for a nice overview); the existence of such controllers is characterized in terms of the existence of structured solutions X, Y to a pair of LMIs subject to an additional coupling condition

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0. \tag{1.1}$$

Moreover, the rank of the various components of the controller state-space can be prescribed by imposing additional rank conditions on $\begin{bmatrix} X & I \\ I & Y \end{bmatrix}$.

The purpose of this paper is to explain the precise logical connections between results on robust stabilization versus results on robust performance. One direction is straightforward: *any result on robust performance gives rise to a result on robust stabilization by specializing the robust performance result to the case where the disturbance and error channels are trivial.* To recover the precise form of the already existing results on robust stabilization however often requires some additional algebraic manipulation. The converse direction is less obvious: *any result on robust stabilization implies a result on robust performance.* In its simplest form, as pointed out in [17], this is a consequence of the Main Loop Theorem for linear-fractional maps (see [22, Theorem 11.7, page 284]):

Principle of Reduction of Robust Performance to Robust Stabilization: *robust performance can be reduced to robust stability by adding a (fictitious) full-block uncertainty feedback connection from the error channel to the disturbance channel.*

The main point of the present paper is that this reduction of robust performance to robust stability is not explained precisely in the literature for the case of dynamic feedback for multidimensional systems. If one casually applies this principle to the result from [17] for the dynamic-feedback case, one arrives at the results from [18, 1] for robust performance, but without the additional coupling constraint (1.1). The explanation is that the condition with the coupling constraint dropped does solve a robust-performance problem, which, however, is a contrived problem of no physical interest, namely the feedback configuration as on the left side of Figure 1: an LFT model Σ for structured uncertainty Δ with a controller Σ_K that, besides the controller-structured uncertainty Δ_K, is granted access to the artificial full-block uncertainty Δ_{full} that connects the error channel y_1 with the disturbance

channel u_1. Instead one must insist that the controller partial-state dimensions for the states corresponding to the artificial full-block uncertainty are zero so that the feedback configuration is as on the right side of Figure 1. This additional constraint on the dimension of the associated block of the controller state space leads to the missing coupling condition. In this way the reduction of robust performance to robust stabilization does hold, but with proper attention paid to the controller information structure. Clarification of this point is the main contribution of the present paper.
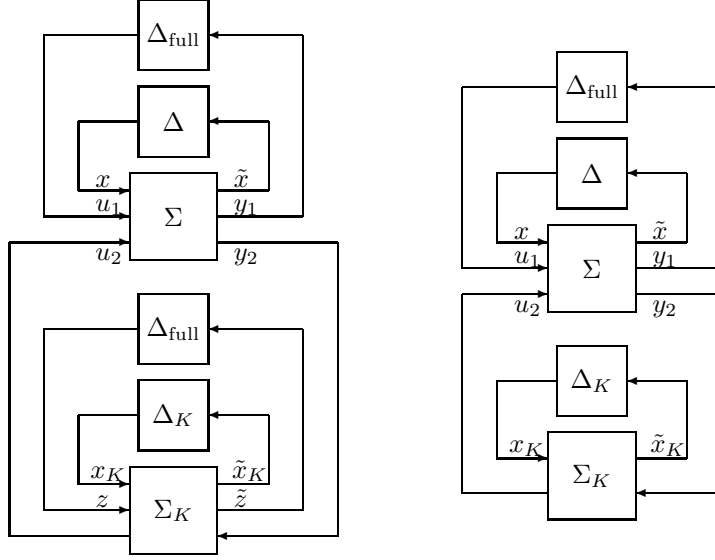

FIGURE 1. Controller with and without access to the full block

We mention that another approach to robust control is to design a controller to guarantee a uniform bound on the L^∞-gain of the disturbance-to-error map (L^1-control) rather than a uniform bound on the L^2-gain of the disturbance-to-error map (H^∞-control); a good overview for L^1-control is the book [8]. However, our focus here is on H^∞-control with the added feature that the disturbance/uncertainty is assumed to have a structured form as given by an LFT model.

The paper is organized as follows. Section 1 is the present introduction. In Section 2 we review some known results concerning LFT-model systems and $\mathcal{Q}$-stability and $\mathcal{Q}$-performance via output feedback. In particular, we recall a theorem from [10] on $\mathcal{Q}$-performance via output feedback with controller partial-state dimension bounds. In the third section we observe how this $\mathcal{Q}$-performance result can be applied to obtain a result on $\mathcal{Q}$-stabilizability with dimension bounds on the controller; in the two extreme cases where either (1) one demands that the controller be static or (2) one imposes no restrictions on the size of the partial states of the controller, auxiliary coupling conditions in the criterion can be eliminated and we recover two $\mathcal{Q}$-stabilizability results from [17] as corollaries. In Section 4 we show that our theorem on $\mathcal{Q}$-stabilizability is actually equivalent to the $\mathcal{Q}$-performance result in Section 2. We conclude this paper with a section on applications for systems with structured uncertainty and for LPV systems.

2. LFT-MODEL SYSTEMS IN GENERAL

Let $\mathbb{F}$ be a field, taken to be either the complex numbers $\mathbb{C}$ or the real numbers $\mathbb{R}$. We define an *LFT model for structured uncertainty* as follows. Assume that the state space $\mathcal{X}$, the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ are all finite-dimensional vector spaces over $\mathbb{F}$, say

$$\mathcal{X} = \mathbb{F}^Z, \quad \mathcal{U} = \mathbb{F}^M, \quad \mathcal{Y} = \mathbb{F}^N.$$

We then specify a direct-sum decomposition for $\mathcal{X}$

$$\mathcal{X} = \oplus_{k=1}^d \mathcal{X}_k \text{ with } \mathcal{X}_k = \mathbb{F}^{n_k \cdot m_k}, \, n_1 \cdot m_1 + \cdots + n_d \cdot m_d = Z. \qquad (2.1)$$

with associated *uncertainty structure* to be the collection $\mathbf{\Delta}$ of matrices of the block-diagonal form

$$\mathbf{\Delta} = \left\{ \Delta = \begin{bmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_d \end{bmatrix} : \Delta_k = \begin{bmatrix} \delta_{k,11}I_{n_k} & \cdots & \delta_{k,1m_k}I_{n_k} \\ \vdots & & \vdots \\ \delta_{k,m_k1}I_{n_k} & \cdots & \delta_{k,m_km_k}I_{n_k} \end{bmatrix} \right\}, \qquad (2.2)$$

where $\delta_{k,ij}$ are arbitrary complex numbers for $k = 1, \ldots, d$ and $i, j = 1, \ldots, m_k$. For short let us use the abbreviation

$$\begin{bmatrix} \delta_{k,11}I_{n_k} & \cdots & \delta_{k,1m_k}I_{n_k} \\ \vdots & & \vdots \\ \delta_{k,m_k1}I_{n_k} & \cdots & \delta_{k,m_km_k}I_{n_k} \end{bmatrix} =: \Delta_k^0 \otimes I_{n_k}$$

where we have introduced the $m_k \times m_k$ matrix Δ_k^0 with scalar entries given by

$$\Delta_k^0 = \begin{bmatrix} \delta_{k,11} & \cdots & \delta_{k,1m_k} \\ \vdots & & \vdots \\ \delta_{k,m_k1} & \cdots & \delta_{k,m_km_k} \end{bmatrix}. \qquad (2.3)$$

Then we define an *LFT model* (for structured uncertainty) to be any collection of the form

$$\left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}, \quad \mathbf{\Delta} \right\}.$$

We shall also have use of the commutant of $\mathbf{\Delta}$, denoted as $\mathcal{D}_{\mathbf{\Delta}}$:

$$\mathcal{D}_{\mathbf{\Delta}} = \{X \in \mathcal{L}(\mathcal{X}) : X\Delta = \Delta X \text{ for all } \Delta \in \mathbf{\Delta}\}.$$

Explicitly, one can show that the commutant $\mathcal{D}_{\mathbf{\Delta}}$ consists of matrices Q of the $d \times d$-block diagonal form

$$Q = \begin{bmatrix} Q_1 & & \\ & \ddots & \\ & & Q_d \end{bmatrix} \qquad (2.4)$$

where, for $k = 1, \ldots, d$, the k-th diagonal entry Q_k in turn has the $m_k \times m_k$-block repeated diagonal form

$$Q_k = \begin{bmatrix} Q_{k,0} & & \\ & \ddots & \\ & & Q_{k,0} \end{bmatrix} \qquad (2.5)$$

where, finally, the repeated block $Q_{k,0}$ is an arbitrary matrix of size $n_k \times n_k$ with scalar entries.

The associated *transfer function* in this context is the function of $\Delta \in \mathbf{\Delta}$ (defined at least for Δ having sufficiently small norm) given as the associated upper linear fractional transformation with symbol $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and load Δ:

$$\mathcal{F}_u \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta \right) = D + C(I - \Delta A)^{-1} \Delta B \in \mathcal{L}(\mathcal{U}, \mathcal{Y}). \tag{2.6}$$

Occasionally we shall also have use for the associated lower linear fractional transformation with symbol $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and load Δ':

$$\mathcal{F}_\ell \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \Delta' \right) = A + B(I - \Delta' D)^{-1} \Delta' C \in \mathcal{L}(\mathcal{X}).$$

This abstract notion of LFT model is used in [17] (see the references there for more background) to model linear input/state/output systems having structured uncertainty.

The classical case corresponds to the case $\mathbf{\Delta} = \{\lambda I_\mathcal{X}\}$ with $\lambda \in \mathbb{C}$ where λ is the frequency variable; in this case, the upper linear fractional transformation is the transfer function of the discrete-time input/state/output linear system

$$\begin{cases} x(n+1) & = & Ax(n) + Bu(n) \\ y(n) & = & Cx(n) + Du(n) \end{cases} \quad n = 0, 1, 2, \ldots \tag{2.7}$$

in the sense that $\mathcal{F}_u \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right], \lambda I_\mathcal{X} \right)$ is the Z-transform

$$\mathcal{F}_u \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right], \lambda I_\mathcal{X} \right) = D + \sum_{n=1}^{\infty} C A^{n-1} B \lambda^n$$

of the impulse response $\{D, CB, CAB, \ldots, CA^{n-1}B, \ldots\}$ of the system (2.7), i.e., the output generated from zero initial condition and input signal corresponding to the unit impulse at time 0 ($u(0) = I_\mathcal{U}, u(n) = 0$ for $n > 0$). In case $\mathbf{\Delta}$ has the form (2.2) with $\mathbb{F} = \mathbb{C}$ and $n_k = 1$ for all k, then $\mathcal{F}_u \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right], \Delta \right)$ can similarly be interpreted as the transfer function of a multidimensional linear system of Givone-Roesser type evolving on the integer lattice: here the frequency variable $\delta = (\delta_1, \ldots, \delta_d) \in \mathbb{C}^d$ is d-dimensional (see [12]). As an alternative interpretation, one can consider $\lambda := \delta_1$ as the frequency variable for a 1-D system and the remaining parameters $\delta_2, \ldots, \delta_d$ as values of parameters specifying a particular choice of disturbance within an admissible set of uncertainties. Then $\mathcal{F}_u \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right], \Delta \right)$, considered as a function of $\lambda = \delta_1$ with the other δ-values $\delta_2, \ldots, \delta_d$ held fixed, specifies the classical transfer function for the system if one assumes the particular choice of uncertainty associated with the given fixed parameter values $\delta_2, \ldots, \delta_d$ (see [18, 1, 10]). One can even let $\delta_1 \ldots, \delta_d$ be formal noncommuting indeterminates and make sense of $\mathcal{F}_u \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right], \Delta \right)$ as a formal power series with coefficients equal to operators from $\mathcal{U}$ to $\mathcal{Y}$; then $\mathcal{F}_u \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right], \Delta \right)$ can be viewed as the transfer function of an input/state/output linear system having evolution along a free semigroup [2]. Alternatively, one can view the diagonal entries $\delta_1, \ldots, \delta_d$ as operators on ℓ^2 with δ_1 equal to the shift operator, interpret $\delta_2, \ldots, \delta_d$ as parameters associated with a particular choice of admissible structured time-varying disturbance in the system, and view the value of $\mathcal{F}_u(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right], \Delta)$ as the input-output map from $\mathcal{U} \otimes \ell^2$ to $\mathcal{Y} \otimes \ell^2$ for the system with particular choice of disturbance specified by the choice of $\delta_2, \ldots, \delta_d$. We discuss some of these various interpretations and their applications in more detail in our final Section 5.

We now recall from [17] how to formulate robust stability and robust performance, along with the related notions of $\mathcal{Q}$-stability and $\mathcal{Q}$-performance, in the general context of an LFT model. Given an LFT model $\Sigma = ([\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}], \boldsymbol{\Delta})$ we make the following definitions:

(1) The LFT model Σ is *robustly stable* if $(I - \Delta A)$ is invertible in $\mathcal{L}(\mathcal{X})$ for all $\Delta \in \overline{\mathcal{B}}\boldsymbol{\Delta} := \{\Delta \in \boldsymbol{\Delta} : \|\Delta\| \leq 1\}$.

(2) The LFT model Σ is *$\mathcal{Q}$-stable* if there exists an invertible $Q \in \mathcal{D}_{\boldsymbol{\Delta}}$ so that $\|Q^{-1}AQ\| < 1$, or, equivalently, if there exists a (strictly) positive-definite $X \in \mathcal{D}_{\boldsymbol{\Delta}}$ (written as $X > 0$) so that $AXA^* - X < 0$.

(3) The LFT model Σ has *robust performance* if Σ is robustly stable and if in addition $\|\mathcal{F}_u([\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}], \Delta)\| < 1$ for all $\Delta \in \overline{\mathcal{B}}\boldsymbol{\Delta}$.

(4) The LFT model Σ has *$\mathcal{Q}$-performance* if there exists an invertible $Q \in \mathcal{D}_{\boldsymbol{\Delta}}$ so that

$$\left\| \begin{bmatrix} Q^{-1} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \right\| < 1, \tag{2.8}$$

or, equivalently, if there exists an $X > 0$ in $\mathcal{D}_{\boldsymbol{\Delta}}$ so that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* - \begin{bmatrix} X & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} < 0. \tag{2.9}$$

Remark 2.1. A couple of remarks are in order to clarify these definitions.

(i) Note that the *robust stability* condition (1) and the *$\mathcal{Q}$-stability* condition (2) involve only the operator $A \colon \mathcal{X} \to \mathcal{X}$. In particular, one can replace the system matrix $[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$ by

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \{0\} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \{0\} \end{bmatrix}$$

without affecting the robust stability or $\mathcal{Q}$-stability of the LFT model.

(ii) Robust performance implies robust stability by definition. It is less obvious but also the case that $\mathcal{Q}$-performance implies $\mathcal{Q}$-stability. Given that Σ has $\mathcal{Q}$-performance, thus given an invertible $Q \in \mathcal{D}_{\boldsymbol{\Delta}}$ satisfying (2.8), it follows in particular that the upper left-hand corner of the matrix inside the norm sign in (2.8) also has norm strictly less than 1, i.e., $\|Q^{-1}AQ\| < 1$, which implies $\mathcal{Q}$-stability.

The following result is well known (see [17, 10]).

Proposition 2.2. *Let $\Sigma := ([\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}], \boldsymbol{\Delta})$ be an LFT-model system. Then the following implications concerning Σ hold:*

(1) *$\mathcal{Q}$-stability $\Longrightarrow$ robust stability.*

(2) *$\mathcal{Q}$-performance $\Longrightarrow$ robust performance.*

Moreover, neither of the implications (1) nor (2) is reversible in general.

For convenience in the discussion to follow, we assume the input space and output space to be of the same finite dimension N, and, in fact, make the identification

$$\mathcal{U} = \mathcal{Y}.$$

The results can be extended to the case $\dim \mathcal{U} \neq \dim \mathcal{Y}$ by using the more general formalism of [3]. We now specify the *full structure* $\boldsymbol{\Delta}_{\text{full}}$ by

$$\boldsymbol{\Delta}_{\text{full}} = \mathcal{L}(\mathcal{U}, \mathcal{Y}) = \mathcal{L}(\mathcal{U}).$$

Note that

$$\mathcal{D}_{\boldsymbol{\Delta}_{\text{full}}} = \{\lambda I_{\mathcal{U}} : \lambda \in \mathbb{F}\}.$$

We shall have use of the structure $\boldsymbol{\Delta} \oplus \boldsymbol{\Delta}_{\text{full}} \subset \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$ consisting of operators of the form

$$\boldsymbol{\Delta} \oplus \boldsymbol{\Delta}_{\text{full}} = \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_0 \end{bmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y}) : \Delta \in \boldsymbol{\Delta}, \, \Delta_0 \in \Delta_{\text{full}} \right\}$$

with associated commutant $\mathcal{D}_{\boldsymbol{\Delta} \oplus \boldsymbol{\Delta}_{\text{full}}}$ given by

$$\mathcal{D}_{\boldsymbol{\Delta} \oplus \boldsymbol{\Delta}_{\text{full}}} = \left\{ \begin{bmatrix} Q & 0 \\ 0 & \lambda I_{\mathcal{U}} \end{bmatrix} : Q \in \mathcal{D}_{\boldsymbol{\Delta}}, \, \lambda \in \mathbb{F} \right\}.$$

Note that the LFT model Σ has $\mathcal{Q}$-*performance* if and only if the system matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]$ is $\mathcal{Q}$-stable with respect to the structure $\boldsymbol{\Delta} \oplus \boldsymbol{\Delta}_{\text{full}}$. Indeed, the condition that $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]$ be $\mathcal{Q}$-stable with respect to $\boldsymbol{\Delta} \oplus \boldsymbol{\Delta}_{\text{full}}$ a priori means that there exist an invertible $Q \in \mathcal{D}_{\boldsymbol{\Delta}}$ and a nonzero number λ so that

$$\left\| \begin{bmatrix} Q^{-1} & 0 \\ 0 & \lambda^{-1} I_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & \lambda I_{\mathcal{U}} \end{bmatrix} \right\| < 1, \tag{2.10}$$

or, equivalently, there exist $X > 0$ in $\mathcal{D}_{\boldsymbol{\Delta}}$ and a number $\mu > 0$ so that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & \mu I_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* - \begin{bmatrix} X & 0 \\ 0 & \mu I_{\mathcal{U}} \end{bmatrix} < 0. \tag{2.11}$$

However we can always replace $\left[\begin{smallmatrix} Q & 0 \\ 0 & \lambda I_{\mathcal{U}} \end{smallmatrix} \right]$ by $\left[\begin{smallmatrix} Q' & 0 \\ 0 & I_{\mathcal{U}} \end{smallmatrix} \right] = \left[\begin{smallmatrix} \lambda^{-1} Q & 0 \\ 0 & I_{\mathcal{U}} \end{smallmatrix} \right]$ in (2.10) and $\left[\begin{smallmatrix} X & 0 \\ 0 & \mu I_{\mathcal{U}} \end{smallmatrix} \right]$ by $\left[\begin{smallmatrix} X' & 0 \\ 0 & I_{\mathcal{U}} \end{smallmatrix} \right] = \left[\begin{smallmatrix} \mu^{-1} X & 0 \\ 0 & I_{\mathcal{U}} \end{smallmatrix} \right]$ in (2.11) to arrive at conditions of the respective forms (2.8) and (2.9). We shall see more of these simplifications via scaling in the sequel.

Robust performance (or $\mathcal{Q}$-performance) can be seen as simply robust stability (respectively, $\mathcal{Q}$-stability) with respect to the appropriately contrived uncertainty structure (see Figure 2), as explained in the following proposition.
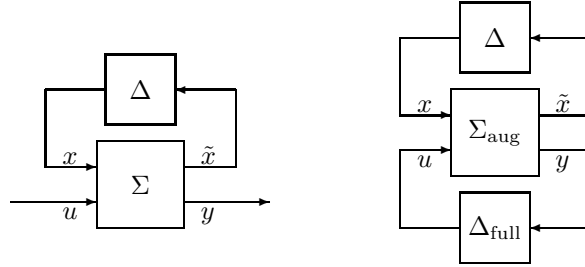

FIGURE 2. Original and augmented LFT-model

Proposition 2.3. *Suppose that we are given an LFT model*

$$\Sigma = \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}, \boldsymbol{\Delta} \right).$$

Form the augmented LFT model Σ_{aug} given by

$$\Sigma_{\text{aug}} = \left(\begin{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \\ \{0\} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \\ \{0\} \end{bmatrix}, \boldsymbol{\Delta} \oplus \boldsymbol{\Delta}_{\text{full}} \right).$$

Then:
  (1) Σ *has robust performance with respect to* $\boldsymbol{\Delta}$ *if and only if* Σ_{aug} *is robustly stable with respect to* $\boldsymbol{\Delta} \oplus \boldsymbol{\Delta}_{\mathrm{full}}$.
  (2) Σ *has* $\mathcal{Q}$*-performance with respect to* $\boldsymbol{\Delta}$ *if and only if* Σ_{aug} *is* $\mathcal{Q}$*-stable with respect to* $\boldsymbol{\Delta} \oplus \boldsymbol{\Delta}_{\mathrm{full}}$.

Proof. See [9, 17]. $\qquad\square$

The general philosophy of feedback control is: *given a plant with deficient properties* (e.g., lack of stability or performance), *design a compensator so that these deficiencies are rectified in the resulting closed-loop system.* To this end, we suppose that we are given an LFT model with input space $\mathcal{U}$ and output space $\mathcal{Y}$ having direct-sum decompositions

$$\mathcal{U} = \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \end{bmatrix}, \qquad \mathcal{Y} = \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix}.$$

Usually the spaces $\mathcal{U}_1, \mathcal{U}_2, \mathcal{Y}_1$ and $\mathcal{Y}_2$ have physical interpretations as disturbance, control, error and measurement signals respectively. Then the LFT model Σ has the more detailed form

$$\Sigma = \left(\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] : \begin{bmatrix} \mathcal{X} \\ \hline \mathcal{U}_1 \\ \mathcal{U}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \hline \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix}, \boldsymbol{\Delta} \right). \tag{2.12}$$

Let us suppose that Σ_K is another LFT model of the form

$$\Sigma_K = \left(\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} : \begin{bmatrix} \mathcal{X}_K \\ \mathcal{Y}_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_K \\ \mathcal{U}_2 \end{bmatrix}, \boldsymbol{\Delta}_K \right). \tag{2.13}$$

Here the uncertainty structure $\boldsymbol{\Delta}_K$ for Σ_K may be independent of the uncertainty structure $\boldsymbol{\Delta}$ for the original LFT model Σ but we will be primarily interested in the case where there is a coupling between $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_K$: we shall give a concrete model for this setup below. In any case, we may form the feedback connection

$$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \widetilde{x} \\ y_1 \\ y_2 \end{bmatrix}, \qquad \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K \\ y_2 \end{bmatrix} = \begin{bmatrix} \widetilde{x}_K \\ u_2 \end{bmatrix}$$

with transfer function $(\Delta, \Delta_K) \mapsto G_{cl}(\Delta, \Delta_K)$ obtained by imposing the additional feedback equations $x = \Delta \widetilde{x}$, $x_K = \Delta_K \widetilde{x}_K$ (see Figure 3).

The resulting closed-loop transfer function is then given by

$$G_{cl}(\Delta, \Delta_K) = \mathcal{F}_\ell \left(\begin{bmatrix} G_{11}(\Delta) & G_{12}(\Delta) \\ G_{21}(\Delta) & G_{22}(\Delta) \end{bmatrix}, \mathcal{F}_u \left(\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}, \Delta_K \right) \right)$$

where

$$\begin{bmatrix} G_{11}(\Delta) & G_{12}(\Delta) \\ G_{21}(\Delta) & G_{22}(\Delta) \end{bmatrix} = \mathcal{F}_u \left(\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right], \Delta \right).$$

As has been observed in [20, 14] and elsewhere (at least for the case where $D_{22} = 0$), one can realize $G_{cl}(\Delta, \Delta_K)$ directly as the transfer function of a linear-fractional model

$$G_{cl}(\Delta, \Delta_K) = \mathcal{F}_u \left(\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}, \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_K \end{bmatrix} \right) \tag{2.14}$$
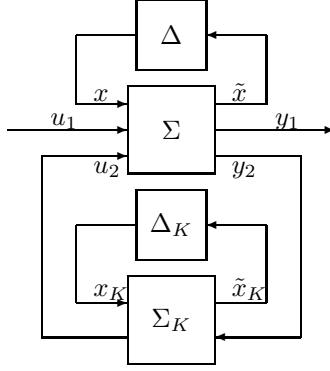
FIGURE 3. Feedback

where the closed-loop state matrix $\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$ is given by

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \mathcal{F}_\ell \left(\left[\begin{array}{ccc|cc} A & 0 & B_1 & 0 & B_2 \\ 0 & 0 & 0 & I & 0 \\ C_1 & 0 & D_{11} & 0 & D_{12} \\ \hline 0 & I & 0 & 0 & 0 \\ C_2 & 0 & D_{21} & 0 & D_{22} \end{array} \right], \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \right). \quad (2.15)$$

The feedback-loop is well-posed exactly when $I - D_{22}D_K$ is invertible. Since, under the assumption of well-posedness, one can always arrange, via a change of variable on the input-output space, that $D_{22} = 0$, it is usually assumed that $D_{22} = 0$; in this case well-posedness is automatic and $\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$ can be written out explicitly as

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \left[\begin{array}{cc|c} A + B_2 D_K C_2 & B_2 C_K & B_1 + B_2 D_K D_{21} \\ B_K C_2 & A_K & B_K D_{21} \\ \hline C_1 + D_{12} D_K C_2 & D_{12} C_K & D_{11} + D_{12} D_K D_{21} \end{array} \right]. \quad (2.16)$$

For the sequel it is convenient to assume $\dim \mathcal{U}_1 = \dim \mathcal{Y}_1$ and identify

$$\mathcal{U}_1 = \mathcal{Y}_1.$$

We set $\mathbf{\Delta}_{\text{full}}$ equal to the *full structure* on $\mathcal{U}_1 = \mathcal{Y}_1$, i.e.,

$$\mathbf{\Delta}_{\text{full}} = \mathcal{L}(\mathcal{U}_1).$$

Given such a pair of LFT models Σ and Σ_K as in (2.12) and (2.13), once we specify a closed-loop structure $\mathbf{\Delta}_{cl}$ we make the following definitions:

(1) The LFT-feedback system (Σ, Σ_K) is *robustly stable* if the closed-loop state matrix A_{cl} is robustly stable with respect to $\mathbf{\Delta}_{cl}$:

$$I - \Delta_{cl} A_{cl} \text{ is invertible for all } \Delta_{cl} \in \overline{\mathcal{B}} \mathbf{\Delta}_{cl}.$$

(2) The LFT-feedback system (Σ, Σ_K) is *$\mathcal{Q}$-stable* if there exists an invertible $Q_{cl} \in \mathcal{D}_{\mathbf{\Delta}_{cl}}$ so that $\|Q_{cl}^{-1} A_{cl} Q_{cl}\| < 1$, or, equivalently, if there exists $X_{cl} \in \mathcal{D}_{\mathbf{\Delta}_{cl}}$ so that $A_{cl} X_{cl} A_{cl}^* - X_{cl} < 0$.

(3) The LFT-feedback system (Σ, Σ_K) has *robust performance* if (Σ, Σ_K) is robustly stable and if in addition the closed-loop transfer function G_{cl} given by (2.14) satisfies

$$\|G_{cl}(\Delta_{cl})\| < 1 \text{ for all } \Delta_{cl} \in \overline{\mathcal{B}}(\mathbf{\Delta}_{cl}). \quad (2.17)$$

(4) The LFT-feedback system (Σ, Σ_K) has $\mathcal{Q}$-*performance* if the closed-loop system matrix $\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$ is $\mathcal{Q}$-stable with respect to the structure $\mathbf{\Delta}_{cl} \oplus \mathbf{\Delta}_{\text{full}}$, i.e., if there exists an invertible $Q_{cl} \in \mathcal{D}_{\mathbf{\Delta}_{cl}}$ so that

$$\left\| \begin{bmatrix} Q_{cl}^{-1} & 0 \\ 0 & I_{\mathcal{U}_1} \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} Q_{cl} & 0 \\ 0 & I_{\mathcal{U}_1} \end{bmatrix} \right\| < 1,$$

or, equivalently, if there exists $X_{cl} > 0$ in $\mathcal{D}_{\mathbf{\Delta}_{cl}}$ so that

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} X_{cl} & 0 \\ 0 & I_{\mathcal{U}_1} \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}^* - \begin{bmatrix} X_{cl} & 0 \\ 0 & I_{\mathcal{U}_1} \end{bmatrix} < 0.$$

As a consequence of Proposition 2.2 and part (3) of Remark 2.1 applied to the closed-loop system, we see that $\mathcal{Q}$-stability for a feedback system (Σ, Σ_K) implies robust stability and that $\mathcal{Q}$-performance implies robust performance (even with $\mathcal{Q}$-stability for the closed-loop system). We also note that the notion of $\mathcal{Q}$-performance for a closed-loop system is equivalent to the controller Σ_K solving the scaled H^∞-problem as formulated in [1] (see Section 5.2 below).

Given an LFT model Σ of the form (2.12), the *robust stabilization problem* is to find an LFT feedback controller Σ_K of the form (2.13) so that the closed-loop system is robustly stable, while the *robust H^∞-problem* is to find Σ_K of the form (2.13) so that the closed-loop system has robust performance. The $\mathcal{Q}$-version of these problems is to find Σ_K which achieves $\mathcal{Q}$-stability and $\mathcal{Q}$-performance, respectively, for the closed-loop system. It happens that necessary and sufficient conditions for the existence of such a Σ_K are only available in general for the $\mathcal{Q}$-version of the problem (see [17], [1]); these same conditions then give sufficient conditions for the non-$\mathcal{Q}$ versions of the problems.

We assume that the controller also has the form of an LFT model. Thus controller state space $\mathcal{X}_K$ has the form

$$\mathcal{X}_K = \mathbb{F}^{Z_K} \text{ and } \mathcal{X}_K = \oplus_{k=1}^d \mathcal{X}_{K,k} \text{ with } \mathcal{X}_{K,k} = \mathbb{F}^{n_{Kk} \cdot m_{Kk}} \tag{2.18}$$

with block structure of the form

$$\mathbf{\Delta}_K = \left\{ \Delta = \begin{bmatrix} \Delta_{K1} & & \\ & \ddots & \\ & & \Delta_{Kd} \end{bmatrix} : \Delta_{K,k} = \Delta_{Kk}^0 \otimes I_{n_{Kk}} \right\}. \tag{2.19}$$

In addition we assume that the structure $\mathbf{\Delta}_{cl}$ for the closed-loop system involves a coupling between the structure for the open-loop plant and that of the controller given by $\Delta_k^0 = \Delta_{Kk}^0$ for $k = 1, \ldots, d$, i.e.,

$$\mathbf{\Delta}_{cl} = \left\{ \begin{bmatrix} \Delta_1^0 \otimes I_{n_1} & & & & & \\ & \ddots & & & & \\ & & \Delta_d^0 \otimes I_{n_d} & & & \\ & & & \Delta_1^0 \otimes I_{n_{K1}} & & \\ & & & & \ddots & \\ & & & & & \Delta_d^0 \otimes I_{n_{Kd}} \end{bmatrix} \right\} \tag{2.20}$$

where the same $m_k \times m_k$ matrix Δ_k^0 appears in the plant block (with multiplicity n_k) and in the controller block (but with multiplicity n_{Kk}). This additional assumption puts no real restriction on the generality of the method, as one can deny the controller (or the LFT system) to have access to certain blocks in the

uncertainty structure simply by setting n_{Kk} (or n_k) equal to zero. If we introduce the permutation matrix P which shuffles the coordinates of the closed-loop state space according to the rule

$$P: \begin{bmatrix} x_1 \\ \vdots \\ x_d \\ x_{K1} \\ \vdots \\ x_{Kd} \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_{K1} \\ \vdots \\ x_d \\ x_{Kd} \end{bmatrix},$$

then the closed-loop structure $\mathbf{\Delta}_{cl}$ in the new coordinates is given by $P^* \mathbf{\Delta}_{cl} P$ and has the same form as (2.2) but with $n_k + n_{Kk}$ in place of n_k. In this representation the associated commutant $\mathcal{D}_{P^* \mathbf{\Delta}_{cl} P}$ therefore has the form (2.4), (2.5) with the index $n_k + n_{Kk}$ in place of n_k.

The definition of robust stability, $\mathcal{Q}$-stability, robust performance and $\mathcal{Q}$-performance we now take with respect to the coupled closed-loop structure given by (2.20). For the rest of the paper we assume that we are given a pair of LFT models (Σ, Σ_K) with this structure.

With these preliminaries out of the way we can state the following precise result.

Theorem 2.4. ($\mathcal{Q}$-performance via dynamic multidimensional output feedback: see Theorem 11.5 in [10]) *There exists a multidimensional dynamic feedback controller Σ_K*

$$\Sigma_K = \left(\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}, \mathbf{\Delta}_K \right)$$

with coupled uncertainty structure as in (2.18) and (2.19) and prescribed controller dimension indices $n_1, \ldots, n_K$ so that the closed-loop system (Σ, Σ_K) (with closed-loop block structure as in (2.20)) has $\mathcal{Q}$-performance if and only if there exist positive-definite matrices $X, Y \in \mathcal{D}_\Delta$ so that

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} AYA^* - Y & AYC_1^* & B_1 \\ C_1 YA^* & C_1 YC_1^* - I & D_{11} \\ B_1^* & D_{11}^* & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0, \tag{2.21}$$

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} A^* XA - X & A^* XB_1 & C_1^* \\ B_1^* XA & B_1^* XB_1 - I & D_{11}^* \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0, \tag{2.22}$$

where N_c and N_o are matrices chosen so that

$$N_c \text{ is injective and } \operatorname{Im} N_c = \operatorname{Ker} \begin{bmatrix} B_2^* & D_{12}^* \end{bmatrix} \text{ and}$$

$$N_o \text{ is injective and } \operatorname{Im} N_o = \operatorname{Ker} \begin{bmatrix} C_2 & D_{21} \end{bmatrix},$$

and, if we write

$$X = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_d \end{bmatrix}, \qquad Y = \begin{bmatrix} Y_1 & & \\ & \ddots & \\ & & Y_d \end{bmatrix}$$

with

$$X_k = \begin{bmatrix} X_{k,0} & & \\ & \ddots & \\ & & X_{k,0} \end{bmatrix}, \qquad Y_k = \begin{bmatrix} Y_{k,0} & & \\ & \ddots & \\ & & Y_{k,0} \end{bmatrix}$$

as in the representation (2.4) *and* (2.5) *for* $\mathcal{D}_{\boldsymbol{\Delta}}$, *then we also have*

$$\begin{bmatrix} X_{k,0} & I \\ I & Y_{k,0} \end{bmatrix} \geq 0 \ and \ \operatorname{rank} \begin{bmatrix} X_{k,0} & I \\ I & Y_{k,0} \end{bmatrix} \leq n_k + n_{Kk} \ for \ k = 1, \dots, d. \quad (2.23)$$

A special case of Theorem 2.4 is the case where one insists that the controller be static, i.e., that all the controller state-space dimensions $n_{K1}, \dots, n_{Kd}$ be equal to 0. In this case, via a Schur-complement argument, one can see that the coupling condition (2.23) assumes the simple form

$$Y_{k,0} = X_{k,0}^{-1} \ for \ k = 1, \dots, d, \quad i.e., \quad Y = X^{-1}. \quad (2.24)$$

We remark that the paper [1] as well as the exposition in the book [10] arrive at Theorem 2.4 directly while the paper [14] (see also [21]), explicitly only for the case $d = 1$ but with an argument extendable to the general case here, first prove the special case for a static controller (conditions (2.21), (2.22) and (2.24)) and then use the observation (2.15) to reduce the dynamic-controller case to the static-controller case.

3. $\mathcal{Q}$-STABILIZATION AS A CONSEQUENCE OF CLOSED-LOOP $\mathcal{Q}$-PERFORMANCE VIA FEEDBACK

By zeroing out the disturbance and error channels, any $\mathcal{Q}$-performance result leads to a $\mathcal{Q}$-stability result. Application of this simple idea to Theorem 2.4 leads to the following $\mathcal{Q}$-stabilization result which we have not seen stated explicitly in the literature.

Theorem 3.1. (*$\mathcal{Q}$-stabilization via dynamic multidimensional output feedback: prescribed controller state-space dimensions*) *There exists a multidimensional dynamic output controller* $\Sigma_K = \left(\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}, \boldsymbol{\Delta}_K \right)$ *(i.e., as in* (2.18) *and* (2.19) *with prescribed dimension indices* $n_{K1}, \dots, n_{Kd}$*) so that the closed-loop system* (Σ, Σ_K) *(with closed-loop block structure* $\boldsymbol{\Delta}_{cl}$ *as in* (2.20)*) is* $\mathcal{Q}$-*stable if and only if there exist positive-definite matrices* $X \in \mathcal{D}_{\boldsymbol{\Delta}}$ *and* $Y \in \mathcal{D}_{\boldsymbol{\Delta}}$ *which satisfy the following pair of LMIs:*

$$B_{\perp}^* A Y A^* B_{\perp} - B_{\perp}^* Y B_{\perp} < 0, \quad (3.1)$$
$$C_{\perp} A^* X A C_{\perp}^* - C_{\perp} X C_{\perp}^* < 0, \quad (3.2)$$

where the matrices $B_{\perp}$ *and* $C_{\perp}$ *are chosen so that*

$$B_{\perp} \ is \ injective \ and \ \operatorname{Im} B_{\perp} = \operatorname{Ker} B_2^*,$$
$$C_{\perp}^* \ is \ surjective \ and \ \operatorname{Im} C_{\perp}^* = \operatorname{Ker} C_2.$$

Here X and Y have the block diagonal form as in (2.4) *and* (2.5)

$$X = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_d \end{bmatrix} \ with \ X_k = \begin{bmatrix} X_{k,0} & & \\ & \ddots & \\ & & X_{k,0} \end{bmatrix}, \quad (3.3)$$

$$Y = \begin{bmatrix} Y_1 & & \\ & \ddots & \\ & & Y_d \end{bmatrix} \ with \ Y_k = \begin{bmatrix} Y_{k,0} & & \\ & \ddots & \\ & & Y_{k,0} \end{bmatrix} \quad (3.4)$$

and must in addition satisfy the coupling and rank conditions

$$\begin{bmatrix} X_{k,0} & I \\ I & Y_{k,0} \end{bmatrix} \geq 0 \ and \ \operatorname{rank} \begin{bmatrix} X_{k,0} & I \\ I & Y_{k,0} \end{bmatrix} \leq n_k + n_{Kk} \ for \ k = 1, \ldots, d. \quad (3.5)$$

Proof. It suffices to apply the observation (i) in Remark 2.1 to the closed-loop system and set the input space $\mathcal{U}_1$ and output space $\mathcal{Y}_1$ equal to $\{0\}$ in Theorem 2.4. Note that in this case the matrices N_c and N_o in Theorem 2.4 coincide with $B_\perp$ and $C_\perp^*$, respectively. $\qquad \square$

There are two extreme special cases of Theorem 3.1: (1) the case where we prescribe $n_{Kk} = 0$ for each $k = 1, \ldots, d$, and (2) the case where no bounds are imposed on n_{Kk}. In each of these cases, the coupling and rank conditions (3.5) either disappear or can be put in a different form. In this way we recover $\mathcal{Q}$-stabilization results appearing in [17] as special cases.

Theorem 3.2. (1) $\mathcal{Q}$-stabilization via static output feedback (see Theorem III-9 in [17]:) *There exists a static output feedback controller (i.e., $\Sigma_K = (D_K, 0)$ where $n_{Kk} = 0$ for $k = 1, \ldots, d$) so that the closed-loop system (Σ, Σ_K) is $\mathcal{Q}$-stable if and only if there exists a positive-definite matrix $X \in \mathcal{D}_\Delta$ so that the following two LMIs hold:*

$$B_\perp^* A X^{-1} A^* B_\perp - B_\perp^* X^{-1} B_\perp < 0, \quad (3.6)$$

$$C_\perp A^* X A C_\perp^* - C_\perp X C_\perp^* < 0. \quad (3.7)$$

Here the matrices $B_\perp$ and $C_\perp$ are chosen as in Theorem 3.1.

(2) $\mathcal{Q}$-stabilization via dynamic multidimensional output feedback (see Theorem V-1 in [17]): *There exists a multidimensional dynamic feedback controller*

$$\Sigma_K = \left(\begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}, \boldsymbol{\Delta}_K \right)$$

(i.e., as in (2.18) and (2.19) with no restriction on the dimension indices $n_{K1}, \ldots, n_{Kd}$) so that the closed-loop system (Σ, Σ_K) (with closed-loop block structure $\boldsymbol{\Delta}_{cl}$ as in (2.20)) is $\mathcal{Q}$-stable if and only if there exist positive-definite matrices $X \in \mathcal{D}_\Delta$ and $Y \in \mathcal{D}_\Delta$ which satisfy the following pair of LMIs:

$$AYA^* - Y - B_2 B_2^* < 0, \quad (3.8)$$

$$A^* X A - X - C_2^* C_2 < 0. \quad (3.9)$$

Proof. To prove the first statement, apply Theorem 3.1 to the case where $n_{K1} = \cdots = n_{Kd} = 0$. Note that the conditions (3.1) and (3.2) are exactly conditions (3.6) and (3.7) but with Y taken to be equal to X^{-1}. Note also that the rank condition

$$\operatorname{rank} \begin{bmatrix} X_{k,0} & I \\ I & Y_{k,0} \end{bmatrix} = n_k$$

(where $n_k = \operatorname{rank} X_{k,0} = \operatorname{rank} Y_{k,0}$ for each k) is equivalent to $X_{k,0} = Y_{k,0}^{-1}$ for each k, or $X = Y^{-1}$.

To prove the second statement, apply Theorem 3.1 to the case where there are no restrictions on the dimension indices $n_{K1}, \ldots, n_{Kd}$. Let $B_\perp$ and $C_\perp$ be as in Theorem 3.1. As pointed out by one of the reviewers, the existence of positive-definite $X, Y \in \mathcal{D}_\Delta$ satisfying (3.8) and (3.9) is equivalent to existence of (not necessarily the same) positive-definite $X, Y \in \mathcal{D}_\Delta$ satisfying (3.2) and (3.1), by a simple application of Finsler's lemma (see [14, Lemma 3]).

As we are imposing no constraints on the control state-space dimension indices $n_{K1}, \ldots, n_{Kd}$, the rank conditions in (3.5) can safely be ignored. To handle the coupling conditions

$$\begin{bmatrix} X_{k,0} & I \\ I & Y_{k,0} \end{bmatrix} \geq 0, \tag{3.10}$$

note that we can always replace $X > 0$ and $Y > 0$ by $\widetilde{X} = \mu X$, $\widetilde{Y} = \mu Y$ with the scalar multiplier $\mu > 0$ sufficiently large to guarantee (3.10) (with $\widetilde{Y}, \widetilde{X}$ in place of Y, X) while not affecting the validity of the homogeneous LMIs (3.1) and (3.2).   □

4. Closed-loop $\mathcal{Q}$-performance as a consequence of $\mathcal{Q}$-stabilization

In this section we give two illustrations of the Principle of Reduction of Robust Performance to Robust Stabilization given in the introduction. Proposition 2.3 is one such illustration, but note that Proposition 2.3 pays no heed to compensator partial-state dimension. Application of the idea in Proposition 2.3 to Theorem 3.2 (2) leads to the following result.

Theorem 4.1. ($\mathcal{Q}$-performance via a dynamic output controller with input-output-loop dynamics) *There exists a multidimensional dynamic output feedback-controller as in the configuration on the left side of Figure 1 which achieves $\mathcal{Q}$-performance for the closed-loop system if and only if there exist positive-definite matrices $X, Y \in \mathcal{D}_{\boldsymbol{\Delta}}$ which satisfy the LMIs (2.21) and (2.22).*

Remark 4.2. We emphasize that the feedback configuration on the left side of Figure 1 is contrived and not of interest from the physical point of view. The point here is that adherence to the Principle of Reduction of Robust Performance to Robust Stabilization does give the equivalence between two control problems, but sometimes not between problems of practical interest, contrary to expectations as suggested in [17].

Proof. Let Σ be the LFT model (2.12) and Σ_K the LFT model (2.13). By definition, the closed-loop LFT-feedback system $\Sigma_{cl} = (\Sigma, \Sigma_K)$ has $\mathcal{Q}$-performance if the closed-loop system matrix $\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$ in (2.16) is $\mathcal{Q}$-stable.

Next we introduce the adjusted LFT model Σ_{adj} given by

$$\Sigma_{adj} = \left(\left[\begin{array}{cc|cc} A & B_1 & 0 & B_2 \\ C_1 & D_{11} & 0 & D_{12} \\ \hline 0 & 0 & 0 & 0 \\ C_2 & D_{21} & 0 & 0 \end{array} \right] : \begin{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U}_1 \end{bmatrix} \\ \{0\} \\ \mathcal{U}_2 \end{bmatrix} \to \begin{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U}_1 \end{bmatrix} \\ \{0\} \\ \mathcal{U}_2 \end{bmatrix}, \boldsymbol{\Delta} \oplus \boldsymbol{\Delta}_{\text{full}} \right), \tag{4.1}$$

and its closed-loop LFT model $\Sigma_{adj,cl} = (\Sigma_{adj}, \Sigma_K)$.

We claim that Σ_{cl} has $\mathcal{Q}$-performance if and only if $\Sigma_{adj,cl}$ is $\mathcal{Q}$-stable. To see this, note that the state operator $A_{adj,cl}$ for the LFT model $\Sigma_{adj,cl}$ is given by

$$\begin{aligned} A_{adj,cl} &= \begin{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} D_K \begin{bmatrix} C_2 & D_{2,1} \end{bmatrix} & \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} C_K \\ B_K \begin{bmatrix} C_2 & D_{21} \end{bmatrix} & A_K \end{bmatrix} \\ &= \begin{bmatrix} A + B_2 D_K C_2 & B_1 + B_2 D_K D_{21} & B_2 C_K \\ C_1 + D_{12} D_K C_2 & D_{11} + D_{12} D_K D_{21} & D_{12} C_K \\ B_K C_2 & B_K D_{21} & A_K \end{bmatrix}. \end{aligned}$$

By rearranging rows and columns we can identify $A_{adj,cl}$ with the closed-loop system matrix $\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$ as in (2.16); in particular, it follows that $A_{adj,cl}$ is $\mathcal{Q}$-stable if and only if $\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}$ is $\mathcal{Q}$-stable as claimed.

Applying Theorem 3.2 (2) to the adjusted LFT model Σ_{adj} thus provides us with necessary and sufficient conditions for the existence of a multidimensional feedback controller Σ_K so that the closed-loop system Σ_{cl} has $\mathcal{Q}$-performance and that has access to $\boldsymbol{\Delta}_K$ in (2.19) as well as to the full block $\boldsymbol{\Delta}_{\text{full}}$.

As was already remarked in the proof of Theorem 3.2, as a consequence of the Finsler lemma the LMIs (3.8), (3.9) are equivalent to the LMIs (3.1), (3.2). It thus remains to show that the LMIs (3.1) and (3.2) when specified to Σ_{adj} are equivalent to the LMIs (2.21) and (2.22). Notice that the matrices $B_\perp$ and $C_\perp$, when specified for Σ_{adj} rather than for Σ, coincide with N_c and N_o^* in Theorem 2.4. For the record we note that (3.1) and (3.2), spelled out for the case at hand, assume the form

$$N_c^* \left(\begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & \mu I \end{bmatrix} \begin{bmatrix} A^* & C_1^* \\ B_1^* & D_{11}^* \end{bmatrix} - \begin{bmatrix} Y & 0 \\ 0 & \mu I \end{bmatrix} \right) N_c < 0, \qquad (4.2)$$

$$N_o^* \left(\begin{bmatrix} A^* & C_1^* \\ B_1^* & D_{11}^* \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & \widetilde{\mu} I \end{bmatrix} \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & \widetilde{\mu} I \end{bmatrix} \right) N_o < 0 \qquad (4.3)$$

with N_c and N_o as in Theorem 2.4. As these inequalities are homogeneous in $\begin{bmatrix} Y & 0 \\ 0 & \mu I \end{bmatrix}$ and $\begin{bmatrix} X & 0 \\ 0 & \widetilde{\mu} I \end{bmatrix}$ respectively, at this stage we may rescale if necessary to arrange without loss of generality that $\mu = \widetilde{\mu} = 1$. Theorem 4.1 follows once we see that conditions (4.2) and (4.3) can be converted to the more linear form of conditions (2.21) and (2.22).

But this last step is a standard Schur-complement computation. We will show only that (2.21) is equivalent to (4.2) as the equivalence of (2.22) with (4.3) is similar. Rewrite (2.21) in the form

$$\begin{bmatrix} N_c^* \begin{bmatrix} AYA^* - Y & AYC_1^* \\ C_1YA^* & C_1YC_1^* - I \end{bmatrix} N_c & N_c^* \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} \\ \begin{bmatrix} B_1^* & D_{11}^* \end{bmatrix} N_c & -I \end{bmatrix} < 0.$$

Validity of (2.21) is equivalent to negative definiteness of the Schur complement with respect to the lower right entry $-I$:

$$0 > N_c^* \begin{bmatrix} AYA^* - Y & AYC_1^* \\ C_1YA^* & C_1YC_1^* - I \end{bmatrix} N_c + N_c^* \begin{bmatrix} B_1 \\ D_{11} \end{bmatrix} \begin{bmatrix} B_1^* & D_{11}^* \end{bmatrix} N_c$$

$$= N_c^* \begin{bmatrix} AYA^* - Y + B_1 B_1^* & AYC_1^* + B_1 D_{11}^* \\ C_1YA^* + D_{11}B_1^* & C_1YC_1^* - I + D_{11}D_{11}^* \end{bmatrix} N_c$$

which, upon rearrangement, agrees with (4.2) (with $\mu = 1$) as expected. This completes the proof of Theorem 4.1. $\qquad \square$

We now show how imposing the condition that the controller state-space dimension constraint $n_{K,\mathcal{U}_1} = 0$ (see the right signal-flow diagram in Figure 1) leads to a proof of Theorem 2.4 on $\mathcal{Q}$-performance as a consequence of the result of Theorem 3.1 on $\mathcal{Q}$-stability; there follows a presumably new interpretation of the coupling condition (2.23) in Theorem 2.4 as the precise extra condition required in Theorem 4.1 for the existence of a controller Σ_K as in Theorem 4.1 which does not have access to the artificial full-block $\boldsymbol{\Delta}_{\text{full}}$.

Theorem 4.3. *The $\mathcal{Q}$-performance result Theorem 2.4 can be seen as a corollary to the $\mathcal{Q}$-stabilization result Theorem 3.1.*

Proof. We follow the same scheme as used in the proof of Theorem 4.1 above but now with use of Theorem 3.1 rather than Theorem 3.2 (2) and with the imposition of the constraint that the controller has no access to the artificial full block $\mathbf{\Delta}_{\mathrm{full}}$. For the special situation where $\Sigma = \Sigma_{\mathrm{adj}}$ as in (4.1), conditions (3.1) and (3.2) in Theorem 3.1 become the LMIs (4.2) and (4.3) given above combined with the two coupling and rank conditions

$$\begin{bmatrix} X_{k,0} & I \\ I & Y_{k,0} \end{bmatrix} \geq 0 \text{ and } \mathrm{rank} \begin{bmatrix} X_{k,0} & I \\ I & Y_{k,0} \end{bmatrix} \leq n_k + n_{Kk} \text{ for } k = 1, \ldots, d, \quad (4.4)$$

$$\begin{bmatrix} \mu I_{\mathcal{U}_1} & I_{\mathcal{U}_1} \\ I_{\mathcal{U}_1} & \widetilde{\mu} I_{\mathcal{U}_1} \end{bmatrix} \geq 0 \text{ and } \mathrm{rank} \begin{bmatrix} \mu I_{\mathcal{U}_1} & I_{\mathcal{U}_1} \\ I_{\mathcal{U}_1} & \widetilde{\mu} I_{\mathcal{U}_1} \end{bmatrix} = \dim \mathcal{U}_1, \quad (4.5)$$

(where X and Y are given by the block diagonal forms as in (3.3) and (3.4)).

As in the proof of Theorem 4.1 we would like to rescale $\begin{bmatrix} Y & 0 \\ 0 & \mu I \end{bmatrix}$ and $\begin{bmatrix} X & 0 \\ 0 & \widetilde{\mu} I \end{bmatrix}$ in (4.2) and (4.3) with μ^{-1} and $\widetilde{\mu}^{-1}$, respectively, to obtain $\mu = \widetilde{\mu} = 1$ and arrive at the equivalence of conditions (3.1), (3.2) with conditions (2.21), (2.22). First we need to check that this rescaling does not violate the coupling and rank conditions in (4.4) and (4.5).

The rank condition in (4.5) forces

$$\widetilde{\mu} = 1/\mu,$$

and conversely, having $\widetilde{\mu} = 1/\mu$ the coupling and rank condition in (4.5) are satisfied. Hence (4.5) is equivalent to $\widetilde{\mu} = 1/\mu$; we therefore assume for the remainder of the proof that $\widetilde{\mu} = 1/\mu$.

In particular, we may rescale $\begin{bmatrix} Y & 0 \\ 0 & \mu I \end{bmatrix}$ and $\begin{bmatrix} X & 0 \\ 0 & \widetilde{\mu} I \end{bmatrix}$ without violating (4.5), but not independently: when we rescale $\begin{bmatrix} Y & 0 \\ 0 & \mu I \end{bmatrix}$ with $\alpha > 0$, then $\begin{bmatrix} X & 0 \\ 0 & \widetilde{\mu} I \end{bmatrix}$ should be rescaled with α^{-1}. In particular, taking $\alpha = \widetilde{\mu}$, given that $\widetilde{\mu} = 1/\mu$, leads to the desired result; after a rescaling we may take $\mu = \widetilde{\mu} = 1$ and maintain the validity of (4.2), (4.3) and (4.5).

It remains to see whether (4.4) still holds under this rescaling. To verify this, observe that

$$\begin{bmatrix} \widetilde{\mu}^{-1} X_{k,0} & I \\ I & \widetilde{\mu} Y_{k,0} \end{bmatrix} = \begin{bmatrix} \widetilde{\mu}^{-1} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_{k,0} & I \\ I & Y_{k,0} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \widetilde{\mu} I \end{bmatrix}$$
$$= \widetilde{\mu}^{-1} \begin{bmatrix} I & 0 \\ 0 & \widetilde{\mu} I \end{bmatrix} \begin{bmatrix} X_{k,0} & I \\ I & Y_{k,0} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \widetilde{\mu} I \end{bmatrix}.$$

We thus obtain that the conditions (4.2), (4.3), (4.4) and (4.5) are exactly equivalent to the conditions (2.21), (2.22) and (2.23) given in Theorem 2.4, and we arrive at the $\mathcal{Q}$-performance result Theorem 2.4 as a consequence of the $\mathcal{Q}$-stabilization result Theorem 3.1 as asserted. $\square$

5. Applications

In this section we discuss how the abstract results on LFT model systems of the previous section apply to more concrete control settings. We discuss two particular applications: *robust control for systems with structured uncertainty* and *robust control for LPV systems*.

5.1. **Systems with LFT models for structured uncertainty.** We suppose that we are given a standard linear time-invariant input/state/output linear system model

$$\Sigma: \begin{cases} x(t+1) &= A_M(\delta_U)x(t) + B_{M1}(\delta_U)w(t) + B_{M2}(\delta_U)u(t) \\ z(t) &= C_{M1}(\delta_U)x(t) + D_{M11}(\delta_U)w(t) + D_{M12}(\delta_U)u(t) \\ y(t) &= C_{M2}(\delta_U)x(t) + D_{M21}(\delta_U)w(t) + D_{M22}(\delta_U)u(t) \end{cases}$$

where the system matrix

$$\begin{bmatrix} A_M(\delta_U) & B_{M1}(\delta_U) & B_{M2}(\delta_U) \\ C_{M1}(\delta_U) & D_{M11}(\delta_U) & D_{M12}(\delta_U) \\ C_{M2}(\delta_U) & D_{M21}(\delta_U) & D_{M22}(\delta_U) \end{bmatrix}$$

is not known exactly but depends on some *uncertainty* parameters $\delta_U = (\delta_1, \ldots, \delta_d)$. Here the quantities δ_i are viewed as uncertainties unknown to the controller. The goal is to design a controller Σ_K (independent of δ_U) so that the closed-loop system has desirable properties for all admissible values of δ_U, usually normalized to be $|\delta_k| \leq 1$ for $k = 1, \ldots, d$.

The transfer function for the uncertainty parameter δ_U can be expressed as

$$\begin{aligned} G(\delta_U) &= \begin{bmatrix} D_{M11}(\delta_U) & D_{M12}(\delta_U) \\ D_{M21}(\delta_U) & D_{M22}(\delta_U) \end{bmatrix} \\ &\quad + \lambda \begin{bmatrix} C_{M1}(\delta_U) \\ C_{M2}(\delta_U) \end{bmatrix} (\lambda I_n - A_M(\delta_U))^{-1} \begin{bmatrix} B_{M1}(\delta_U) & B_{M2}(\delta_U) \end{bmatrix} \\ &= \mathcal{F}_u \left(\left[\begin{array}{c|cc} A_M(\delta_U) & B_{M1}(\delta_U) & B_{M2}(\delta_U) \\ \hline C_{M1}(\delta_U) & D_{M11}(\delta_U) & D_{M12}(\delta_U) \\ C_{M2}(\delta_U) & D_{M21}(\delta_U) & D_{M22}(\delta_U) \end{array} \right], \lambda I \right) \end{aligned} \tag{5.1}$$

where we have introduced the aggregate variable

$$\delta = (\delta_U, \lambda) = (\delta_1, \ldots, \delta_d, \lambda).$$

It is not too much of a restriction to assume in addition that the functional dependence on δ_U is given by a linear fractional map (where the subscript U suggests *uncertainty* and the subscript S suggests *shift*)

$$\begin{bmatrix} A_M(\delta_U) & B_{M1}(\delta_U) & B_{M2}(\delta_U) \\ C_{M1}(\delta_U) & D_{M11}(\delta_U) & D_{M12}(\delta_U) \\ C_{M2}(\delta_U) & D_{M21}(\delta_U) & D_{M22}(\delta_U) \end{bmatrix} = \mathcal{F}_u \left(\left[\begin{array}{c|ccc} A_{UU} & A_{US} & B_{U1} & B_{U2} \\ \hline A_{SU} & A_{SS} & B_{S1} & B_{S2} \\ C_{1U} & C_{1S} & D_{11} & D_{12} \\ C_{2U} & C_{2S} & D_{21} & D_{22} \end{array} \right], \Delta_U \right)$$

where we take the uncertainty structure matrix Δ_U to have the form as in (2.2) with $m_k = 1$ for $k = 1, \ldots, d$ for simplicity:

$$\Delta_U = \begin{bmatrix} \delta_1 I_{n_1} & & \\ & \ddots & \\ & & \delta_d I_{n_d} \end{bmatrix}.$$

Finally, if we introduce the aggregate matrix

$$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} A_{UU} & A_{US} & B_{U1} & B_{U2} \\ A_{SU} & A_{SS} & B_{S1} & B_{S2} \\ \hline C_{1U} & C_{1S} & D_{11} & D_{12} \\ C_{2U} & C_{2S} & D_{21} & D_{22} \end{array} \right], \tag{5.2}$$

then the transfer function $G(\delta)$ (5.1) can conveniently be written in LFT form as

$$G(\delta_U) = \mathcal{F}_u\left(\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \Delta\right)$$

where we have now set Δ equal to the expanded block diagonal matrix

$$\Delta = \begin{bmatrix} \Delta_U & 0 \\ 0 & \lambda I \end{bmatrix} \quad (\lambda \in \mathbb{F}).$$

If we take $\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$ as in (5.2) and introduce the block structure

$$\boldsymbol{\Delta} = \left\{ \begin{bmatrix} \delta_1 I_{n_1} & & & \\ & \ddots & & \\ & & \delta_d I_{n_d} & \\ & & & \lambda I_n \end{bmatrix} : \delta_1, \ldots, \delta_d, \lambda \in \mathbb{F} \right\} \tag{5.3}$$

we may consider $(\Sigma, \boldsymbol{\Delta})$ as an LFT model of the form (2.12). Without much loss of generality, we follow the common normalization and assume that $D_{22} = 0$.

The problem is to design an output-feedback controller $K \colon y \mapsto u$ so that the closed-loop system

$$\begin{cases} x(t+1) & = & A_{cl}(\delta_U)x(t) + B_{cl}(\delta_U)w(t) \\ z(t) & = & C_{cl}(\delta_U)x(t) + D_{cl}(\delta_U)w(t) \end{cases}$$

is *robustly stable* (i.e., $A_{cl}(\delta_U)$ has spectral radius less than 1 for all δ_U such that $|\delta_k| \leq 1$ for each $k = 1, \ldots, d$) and, that perhaps also solves the *robust performance* problem, i.e., in addition the closed-loop transfer function $G(\delta)$ satisfies

$$\|G(\delta)\| < 1 \text{ for all } \delta = (\delta_1, \ldots, \delta_d, \lambda) \text{ with } |\delta_k|, |\lambda| \leq 1.$$

If we only allow for static controllers, then a necessary and sufficient condition for a solution to the $\mathcal{Q}$-stabilization problem is given by Theorem 3.2 (1). As $\mathcal{Q}$-stability always implies robust stability, the conditions in Theorem 3.2 (1) give sufficient conditions for the existence of a static controller satisfying the robust stabilization problem.

For the discussion of dynamic controllers some care must be taken, since the quantities $\delta_1, \ldots, \delta_d$ are here uncertainties which are unknown to the controller. To obtain sufficient conditions for the existence of a *dynamic* controller solving the robust stabilization problem, one only needs to apply the more flexible Theorem 2.4 with the prescription that the controller state-space dimensions n_{Kk} are to be equal to 0 for $k = 1, \ldots, d$ but no constraint is imposed on n_{KS} (i.e., the controller is allowed to have dynamics corresponding to the frequency variable λ). Similarly, the conditions in Theorem 3.1 with the imposition that $n_{Kk} = 0$ for $k = 1, \ldots, d$ and $n_{KS} = 0$ (the static controller case) or only $n_{Kk} = 0$ for $k = 1, \ldots, d$ (the case where the controller is allowed to have dynamics with respect to the frequency variable) give sufficient conditions for the existence of a controller which solves the robust performance problem.

As is now well-known (see [19, 17, 3, 10]), if one expands the structured uncertainty to include time-varying structured uncertainty, then robust stability is equivalent to $\mathcal{Q}$-stability and the various conditions in Theorems 3.2, and 3.1 become necessary as well as sufficient for the existence of the respective type of controller

solving the robust stabilization/performance problem. The LFT model for this expanded uncertainty structure amounts to tensoring the system matrix (5.2) with I_{ℓ^2} (the identity operator on the space ℓ^2 of square-summable sequences)

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_2 & \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \otimes I_{\ell^2}$$

and expanding the block structure to have the form

$$\widetilde{\mathbf{\Delta}} = \left\{ \widetilde{\Delta} = \begin{bmatrix} I_{n_1} \otimes \delta_1 \\ & \ddots \\ & & I_{n_d} \otimes \delta_d \\ & & & I_n \otimes S \end{bmatrix} : \delta_k \in \mathcal{L}(\ell^2) \text{ for } k = 1, \ldots, d \right\}$$

(5.4)

where S is the shift operator on ℓ^2

$$S \colon (c_1, c_2, c_3, \ldots) \mapsto (0, c_1, c_2, \ldots).$$

It can be shown that the results are unaffected if one replaces the shift operator S in (5.4) by a general operator $\delta_S \in \mathcal{L}(\ell^2)$; hence the LFT feedback model formalism carries over to this setting.

5.2. **LPV systems.** A second application of LFT models to robust stabilization and performance problems is in the context of gain-scheduling for Linear-Parameter-Varying (LPV) systems. We assume that we are given an LFT model of the form (5.2) and (5.3) where now the quantities $\delta_1, \ldots, \delta_d$ are interpreted to be, rather than uncertainties, plant parameters varying in time. It is assumed that the controller has access to these parameter values $\delta_1, \ldots, \delta_d$ at each point in time t. Then it makes sense to consider robust stabilization and robust performance problems where the controller is allowed to have dynamics in the uncertainty (now parameter) variables as well as in the frequency variable λ. In this setting $\mathcal{Q}$-stability is sufficient but not equivalent to robust stability. We conclude that the conditions in Theorem 3.2 (2) (adapted to the structure (5.2) with (5.3)) are sufficient for the existence of such a "gain-scheduling" controller (see [18]) which achieves robust stability, and, similarly, the conditions of Theorem 3.1 (with constraints on the controller state-space dimensions $n_{K1}, \ldots, n_{Kd}, n_{KS}$ at the discretion of the user) are sufficient for the existence of such a controller achieving robust performance. Theorem 3.1 in this context is one of the main results of the paper [1]; the "scaled-H^∞ problem" defined there is equivalent to finding a controller Σ_K which achieves our "$\mathcal{Q}$-performance" for the closed-loop system.

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